Thrust and Impulse Requirements for Jet Attitude-Control Systems

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The problem of thrust selection for all-jet attitude control systems is analyzed, and a simple analytical approach described. Numerical results are given for three sample vehicles (5, 50, and 500 thousand-pounds weight) in a Mars mission using the described technique. The required thrust level is found to be a function of the propellant used; three distinct thrust levels result when cold gas, hypergolic bipropellant, or monopropellant (using catalytic packs for initiating and sustaining combustion) are compared on the basis of identical impulse requirements. The control-system maneuvers consist of initial vehicle orientation, occasional reorientation and position stabilization during the coast phase, and the midcourse-correction propulsive maneuvers. The results indicate that a single all-jet control system is satisfactory and furthermore that the uniquely defined thrust levels and the associated propellant consumption are low, especially when compared with the vehicle weights.

Nomenclature

α = angular acceleration, rad/sec^2
$\dot{\theta}$ = angular velocity, rad/sec
θ = position angle, rad
δ = deadband angle, rad
t_r = start time lag (from "on" signal to full thrust), sec
t_d = cutoff time lag (from "off" signal to zero thrust), sec
k = switching function slope (lead time), sec
T = torque, ft-lb
J = moment of inertia, ft-lb-sec^2
$\Delta(\;)$ = change in ( )
t = time, sec
m = "time-off" factor in rotation maneuver
L_f = thrust moment arm, ft
I_s = specific impulse, lb-sec/lb
t_{on} = control "on" time, sec
t_{off} = control "off" time, sec
f = "time-off" to "time-on" ratio
F = thrust, lb
$\bar{f}$ = limit cycle frequency, cps
I = impulse, lb-sec
X = total mission time, sec
W_p = weight of propellant, lb
r = control to disturbing torque ratio

Subscripts

0, 1, 2, 3 . . . = conditions on points shown in various sketches
d or D = disturbing (except t_d, q.v. above)
c = control
y = yaw axis
p = pitch axis
r = roll axis
Lc = limit cycle
i = initial
f = final

Introduction

SOME results obtained in a study sponsored by NASA are presented in this paper.[1] The main topic for discussion is a simple approach to the problem of thrust-level selection for all-jet attitude control systems. Analysis is used to calculate precisely the required thrust levels, and the associated impulse and propellant requirements for a number of typical vehicles in a Mars transfer mission.

Although the actual numerical results apply only to the specific cases analyzed, the method of thrust selection can be used for other vehicles and missions, the requirements of which are clearly defined and vary from those used here.

Methods of Analysis

General Approach

An attitude-control system must accomplish initial vehicle acquisition, occasional reorientation, and position stabilization. The first two operations are transient, whereas the latter is the steady-state operation characterized by the existence of a limit cycle. Control-actuation signals are given in terms of either position or velocity errors or both. In the limit-cycle region, control actuation on position error only makes the system unstable unless the control pulse width is fixed, whereas a combination of position and velocity error actuation is stable.[3, 5] A pure time delay is assumed in all command signals before the system responds in a square-wave fashion.

This simplification does not alter the generality of the analysis because impulse, rather than torque, is the important parameter. It results, however, in the elimination of time as an explicit variable of the system equations, making their solution simple through the use of phase-plane techniques.†

All maneuver phase plots are composite curves consisting of parabolas for constant acceleration and straight horizontal lines for constant velocity trajectories. The parabolas are concave to the left or to the right as the control torque becomes negative or positive with the angular acceleration being their semilatus rectum.[2, 3] The difference between the transient and steady-state phase plots is that the latter closes into itself, thus becoming a periodic solution of the system equation (5).[11] This closed phase plot is the limit cycle which can be symmetric or asymmetric, as the steady-state velocity errors oscillate with equal or unequal amplitudes about the zero-error axis; and undisturbed or disturbed, depending respectively upon the absence or presence of external torques.

Thrust Requirements

The analysis of the problem requires knowledge of vehicle angular acceleration. When specific vehicles are considered, however, the acceleration defines the torque because the inertia is considered fixed. In any specific vehicle, the loca-

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† A detailed discussion of the phase plane and its use as a tool for solving nonlinear equations can be found in Refs. 10 and 11.

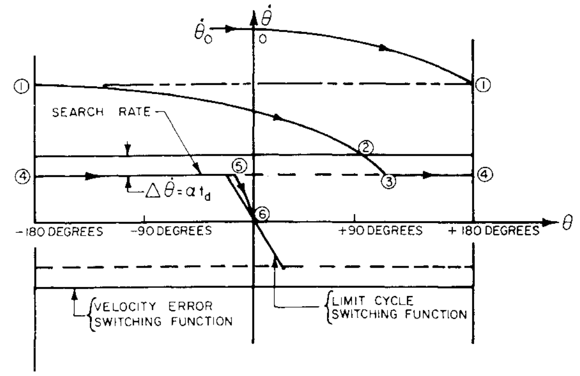

Fig. 1 Acquisition maneuver phase portrait.

tion of the jets is determined by the system geometry and design which implies that the moment arm is fixed and that only the thrust can vary to generate the required control torque.

Acquisition

The purpose of this maneuver is to eliminate the initial velocity errors. The control is actuated on velocity error and keeps operating until a predetermined velocity is attained, at which time cutoff is initiated. The vehicle continues at this low velocity, termed "search rate," and the position sensors have the opportunity to respond and lock onto the correct position; as the vehicle approaches the zero-position error point, the limit-cycle switching function is energized and arrests the motion, reducing all errors to steady-state values. The phase portrait of this maneuver is shown in Fig. 1. The advantages of this mode of operation are 1) no excess impulse is supplied by the control and 2) the limit-cycle switching function is given a slope such that it reduces the search rate to limit-cycle rate with one pulse.

The magnitude of the search rate is set at a level that will permit the position sensors to lock onto the reference position before the vehicle completes one revolution.‡

Equations for this maneuver are:

Propellant weight

$$W_p = J\Delta\dot{\theta}/I_s L_f \tag{1}$$

Angle of rotation

$$\Delta\theta = \dot{\theta}_0^2/2\alpha \tag{2}$$

All systems considered in the study are stable, and if given sufficient time, the maneuver can be accomplished. However, if the requirements specify that the maneuver must be accomplished either within a definite time or that the angular displacement during the maneuver may not exceed a definite value, then the thrust requirements are unique, as shown by the following equations.

Response time (fixed t_{on}):

$$F = J\Delta\dot{\theta}/L_f t_{\text{on}} \tag{3}$$

Angular displacement (fixed $\Delta\theta$):

$$F = \frac{J(\dot{\theta}_f^2 - \dot{\theta}_0^2)}{2L_f \cdot \Delta\theta} \tag{4}$$

where $\dot{\theta}_f$ could be zero.

The phase portraits of various thrust levels used in the reduction of a specific velocity error are shown in Fig. 2. The impulse requirements are only a function of the velocity error $\dot{\theta}_0$ and invariant with thrust level.

‡ Jet Propulsion Laboratory used 720°/hr for the Mariner vehicle.[7]

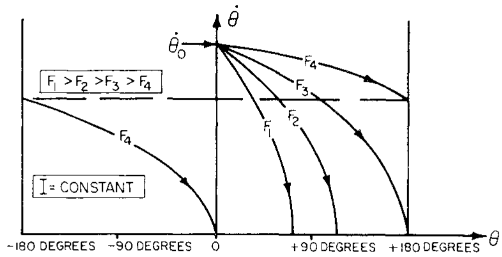

Fig. 2 Acquisition maneuver phase portrait: variation of angular displacement with thrust.

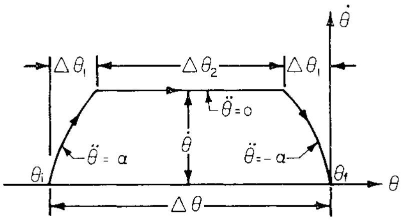

Fig. 3 Rotation maneuver phase portrait.

Rotation

This maneuver rotates the vehicle from one stable position to another within some specified time limit; the initial and final velocities are assumed to be zero. The start command signal is given on a position error and the cutoff is given on velocity. The vehicle then coasts at this constant rate until it approaches the desired position, at which time an equal and opposite control pulse is applied, bringing it to rest or within the limit-cycle region. The phase portrait is shown in Fig. 3. The equations of motion are:

Maneuver velocity

$$\dot{\theta} = \frac{-mt + [m^2t^2 + (4\Delta\theta/\alpha)]^{1/2}}{2/\alpha} \tag{5}$$

Maneuver time

$$t = 2\left[\frac{\Delta\theta}{\alpha(1 - m^2)}\right]^{1/2} \tag{6}$$

Several thrust levels can accomplish this maneuver if only the maneuver magnitude and time are specified; if, in addition, a mean or maximum angular velocity is specified then a unique value of thrust is required. This means that the vehicle must be accelerated in such a manner that a definite velocity increment will be imparted to it within a specified time period. The propellant consumption is not penalized by thrust variation because thrust level and "on-time" vary simultaneously such that their product (the impulse) is a constant. The phase portrait for different thrust levels is shown in Fig. 4.

Position Stabilization

The control system operates in a limit cycle and receives command signals on the basis of both position and velocity errors as indicated by the control law

$$\pm\delta = k\dot{\theta} + \theta \tag{7}$$

The type of limit cycle depends upon the relative magnitude of the disturbing and control torques which in turn have a direct effect on the limit cycle velocity and the associated propellant consumption.

The applicable equations for the disturbed-limit cycle are given in the Appendix. The undisturbed case is described by the equations following.§

§ Details may be found in Refs. 1 and 2.

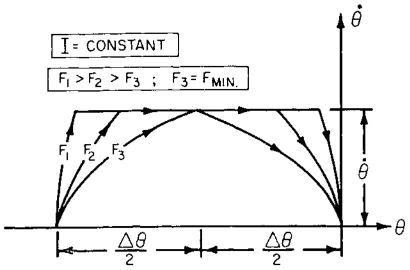

Fig. 4 Rotation maneuver phase portrait for different thrust levels.

Limit cycle velocity and switching point

$$\dot{\theta}_{Lc} = \frac{\alpha t_d(2k - t_d)}{2(2k - t_r - t_d)} \tag{8}$$

$$\theta_{Lc} = \delta - (k - t_r)\dot{\theta}_{Lc} \tag{9}$$

Time "off" to time "on" ratio

$$f = \left(\frac{\alpha}{\dot{\theta}_{Lc}}\right)\left(\frac{\theta_{Lc}}{\dot{\theta}_{Lc}}\right) = \frac{2(2k - t_r - t_d)}{t_d(2k - t_d)}\left[\frac{2(\delta/\alpha)(2k - t_r - t_d)}{t_d(2k - t_d)} - (k - t_r)\right] \tag{10}$$

Propellant consumption

$$Wp = \frac{F}{I_s}\sum_{i=0}^{N}(t_{on})_i = \frac{F}{I_s}\left[\frac{1}{1+f}\right]X \tag{11}$$

If Eq. (10) is differentiated with respect to k and set equal to zero a value of k given by

$$k = \frac{t_0 + [t_0^2 + 32(\delta/\alpha)]^{1/2}}{4} \quad k > 0;\ t_0 = t_r = t_d$$

or

$$k \approx (2\delta/\alpha)^{1/2} \text{ for } k \gg t_0 \tag{12}$$

is obtained which maximizes f, since its minimum value is obviously zero.

These equations clearly indicate that the propellant consumption increases as the square of the thrust. Hence, jets with as low a thrust as possible should be used. The phase portrait of an undisturbed limit cycle for different thrust levels is given in Fig. 5, from which it is seen that increasing thrust increases the frequency and decreases the amplitude of oscillation.

In the disturbed limit cycle, the impulse requirements are invariant with control thrust because they are dictated by the magnitude of the disturbance. The control impulse simply matches the disturbance, and for this reason an absolute minimum thrust level must be available. Use of this thrust level implies continuous operation, and also that the control torque arrests the vehicle at a zero velocity point. This is not a valid assumption, because of the delays between the sensing of the disturbance and the actuation of the control system. The control torque must be higher than the disturbance, in which case a limit cycle will exist. On the other hand, the control thrust can be increased to the point at which the vehicle traverses the entire deadband without any impulse penalty; this is optimum for it and has the lowest pulsing frequency. Thus, the thrust level is bounded between two extremes, without penalizing the system, given by

$$4(T_DJ\delta)^{1/2}/L_ft_d \geq F_c \geq T_D/L_f \tag{13}$$

If the control thrust exceeds these bounds, in the lower case it will not be capable of any control, whereas in the upper case it will penalize the system with regard to impulse by a factor of 3 because the limit cycle will become undisturbed.[4]

Typical phase portraits of the disturbed limit cycle for variable thrust and fixed disturbance are shown in Fig. 6, from which it is seen that increasing thrust increases the amplitude and decreases the frequency of oscillation. Thus, each maneuver by virtue of its peculiar requirements may dictate different thrust levels; the disturbed limit cycle, however, defines a region within which the thrust can vary without affecting the propellant consumption.

In this discussion, thrust levels are selected on the basis of a disturbed limit cycle in the coast phase stabilization maneuver when the vehicles are subjected to small destabilizing solar torques; then their capabilities for accomplishing the other maneuvers are investigated.

Selection of thrust levels

First, the thrust level corresponding to the upper bound for a disturbed limit cycle without any impulse penalty is determined, as outlined in the preceding section. This is given by

$$F_c = 4(T_DJ\delta)^{1/2}/L_ft_d \tag{14}$$

In any specific vehicle the moment of inertia J and the moment arm L_f are constants; the deadband δ is set by the desired tolerances, using the maximum possible; and the disturbing torques are evaluated for the specific mission and system under consideration. This implies that the product of the thrust F_c and the cutoff time lag, i.e., the impulse, is a constant. This, of course, is true because the impulse is set by the disturbance. It follows that any combination of thrust and time lags would meet the requirements; the

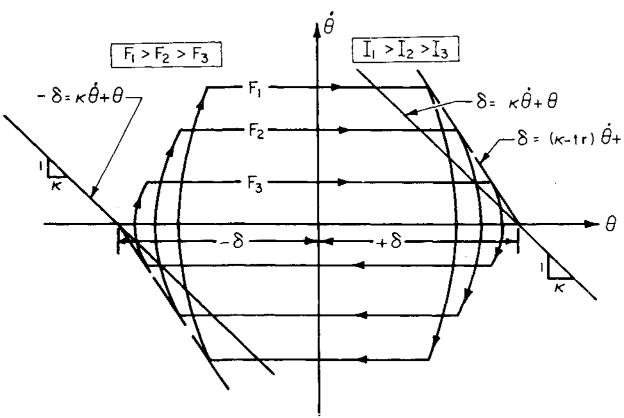

Fig. 5 Undisturbed limit cycle phase portrait for different thrust levels.

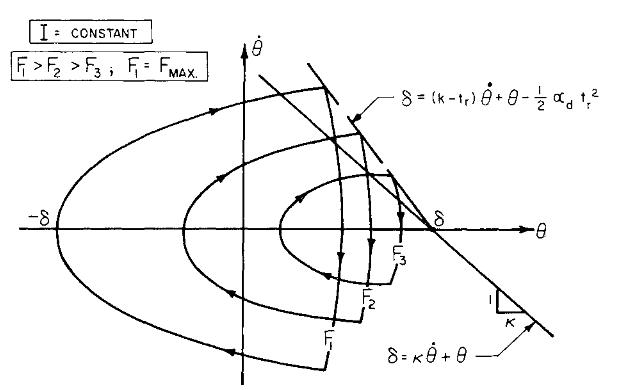

Fig. 6 Disturbed limit cycle portrait of fixed disturbance with variable thrust.

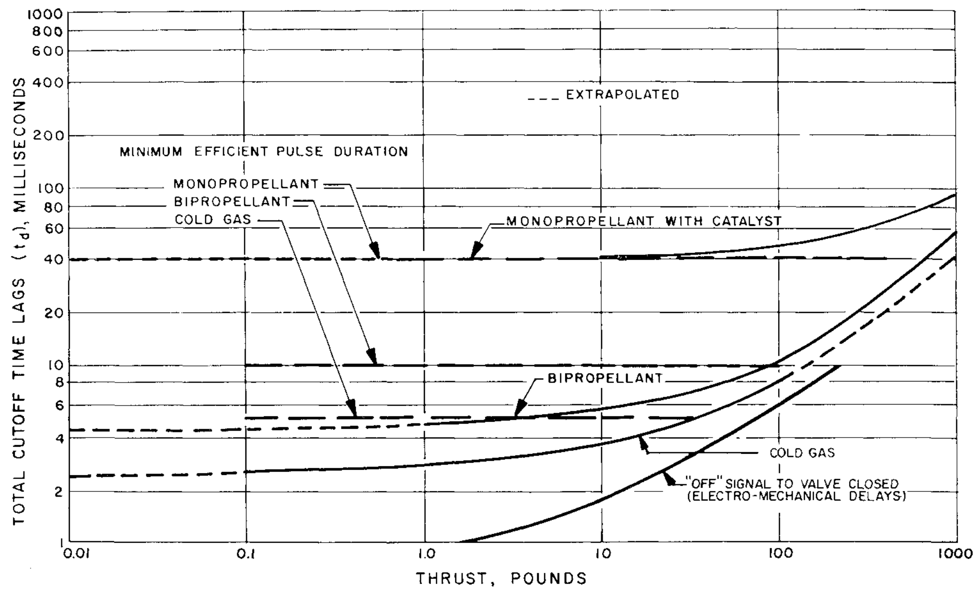

Fig. 7 Variations of system cutoff time lags with thrust.

foregoing equation also indicates that a high thrust would need low time lags, which is in conflict with reality because the time lags usually increase with thrust. The problem, then, is in determining the time lags as a function of thrust, thus eliminating one of the variables from Eq. (14). Propulsion system time lags, however, do not lend themselves to conventional analytic expressions; hence, they were evaluated from experimental data as will be explained below.

Determination of the cutoff time lags

The cutoff time lags consist of electromechanical delays associated with the coil energization and valve closing time, and the thrust decay after propellant flow has ceased. The electromechanical delays depend upon the thrust level because high thrust requires high propellant flows and hence large valves and coils. A plot of experimentally obtained values using operational components is given in Fig. 7 as a function of thrust.

The thrust decay lags are more difficult to evaluate because they depend upon a variety of factors, the most important of which are the propellant volume being trapped downstream of the valve and the exact geometry of the passages.

The effects of propellant combinations are significant when a comparison is made between bipropellants, cold gas, and monopropellant systems. A cold gas system requires no combustion chamber, and the entire system may consist of a valve and a nozzle with the valve seat itself acting as the throat. The bipropellant system requiring combustion chambers and perhaps additional manifolds has higher thrust decay time lags. Finally, the monopropellants may require catalysts which imply larger combustion chamber volumes. Therefore, the cold gas systems have the lowest thrust decay lags, and the monopropellants requiring catalysts have the highest; the bipropellants and the hypergolic-start, self-sustaining monopropellants have intermediate and approximately equal thrust decay time lags.

A survey of existing systems revealed the typical values as shown in Table 1. When these time lags are added to the

Table 1 Propulsion system time lags

System	Thrust decay lag, msec
Cold gas	2
Hypergolic bipropellant	4
Monopropellant requiring catalysts (H_2O_2)	40

electromechanical lags the final curve of total time lags as a function of thrust can be drawn as shown in Fig. 7. The indicated time lags are typical square-wave equivalents, and can be further improved if higher power is available. Present state-of-the-art technology, however, indicates that the minimum efficient square wave pulse widths are 5, 10, and 40 msec for cold gas, bipropellant, and monopropellant systems, respectively. Possible improvements may result in 20-msec pulses for the monopropellant systems.

Thus, a graphic relationship has been established between the thrust level and its associated time lags which can be used in conjunction with Eq. (14) to determine the thrust level precisely. The procedure is to plot t_d vs F_c from Eq. (14) for the particular vehicle under consideration and then on the same graph to superimpose the graphs from Fig. 7. The required thrust levels are uniquely defined by the intersections of the various curves.

Because the impulse is a constant, as indicated by Eq. (14), and because for a given thrust level the time lags (minimum "on-time") are propellant dependent, it follows that the required thrust level will also vary with the propellant used. Cold gas systems having the shortest minimum pulse-width capability must have the highest thrust to supply the required impulse.

Numerical Results

To obtain some numerical values, it is necessary only to define typical vehicles and missions and to evaluate the disturbances.

Three vehicles, whose characteristics are given in Table 2, were selected and placed on a 240-day Mars transfer mission. Two types of disturbances were found to be of significance: solar pressure torques about the vehicle pitch and yaw axes, which are given in Fig. 8, and midcourse correction pro-

Table 2 Sample vehicle data

	Vehicle no.		
	1	2	3
Weight, lb	5000	50,000	500,000
Moments of inertia,[a] lb-ft-sec^2			
Yaw, J_y	3000	175,000	9,900,000
Pitch, J_p	3000	175,000	9,800,000
Roll, J_r	1000	28,000	980,000
Diameter, ft (roll moment arm)	7.4	12	24
Pitch and yaw moment arms, ft	5.1	13	32

[a] Inertias remain constant.

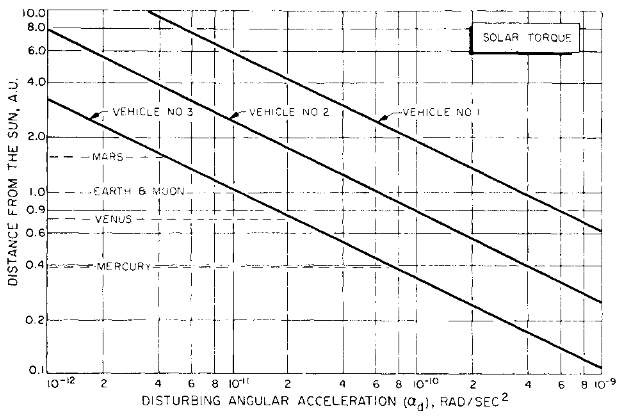

Fig. 8 **Variation of the disturbing angular acceleration with distance from the Sun.**

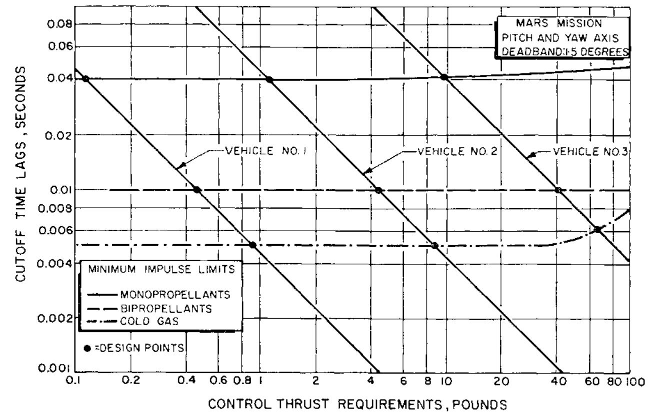

Fig. 10 **Effect of system cutoff time lags on position stabilization thrust requirements.**

pulsion system misalignments given in Fig. 9. During the midcourse maneuver, control about the pitch and yaw axes is accomplished by thrust vector control. Trajectory analysis, found in Ref. 1, indicates that the midcourse correction propulsion system has an optimum thrust level corresponding to a thrust-to-weight ratio of 0.05.

The numerical values of the required upper bound control thrust about the pitch and yaw axes are obtained from Eq. (14) and Fig. 7 as indicated earlier. The results are given in Fig. 10 and Table 3. Because the solar torques are variable for the mission considered, the thrust levels are selected on the basis of their minimum value. This will cause the cycle to be nonoptimum for a large portion of the mission, but disturbed limit-cycle operation is insured. Thus, the limit-cycle frequency will decrease as the mission progresses.

Proceeding in a similar manner, thrust levels could be sized for different maneuvers and for the high disturbing torques caused by the midcourse propulsion system. Instead, the capabilities of the calculated thrust levels in accomplishing the other maneuvers are investigated.

The most critical condition is represented by the system capability for satisfactory roll axis control during the midcourse correction maneuver. The deciding criteria are availability of sufficient control torque and jet pulsing frequency. In the present case the minimum thrust requirements are 0.014, 0.082, and 0.42 lb for vehicles 1, 2, and 3, respectively; the associated pulsing frequencies are 3.0, 2.0, and 1.0 cps, none of which are excessive. Similarly, it can easily be verified that the acquisition and the rotation maneuvers can be accomplished satisfactorily. Thus a single-thrust level system can meet the mission requirements.

Impulse Requirements and Propellant Consumption

As a concluding step, evaluation is made of the total impulse requirements and the associated propellant consumption for the vehicles and mission assumed using the thrust levels calculated. All vehicles are placed through a set of attitude control maneuvers indicated in the following paragraphs.

Acquisition

The control system absorbs all separation rates that are set at 1.15, 0.5, and 0.25 deg/sec for vehicle 1, 2, and 3, respectively, about all three axes.[1]

Position stabilization

This maneuver operates on a disturbed limit cycle with a deadband of 1.5°, on the basis of which the thrust levels about the pitch and yaw axes were selected. Control about the roll axis will be of the undisturbed limit-cycle type because there are no solar torques; the deadband is also 1.5°.

Midcourse correction

1) The vehicle is rotated by 180° about the pitch or yaw axis and 90° about the roll axis in 15 min before the midcourse correction engine firing.

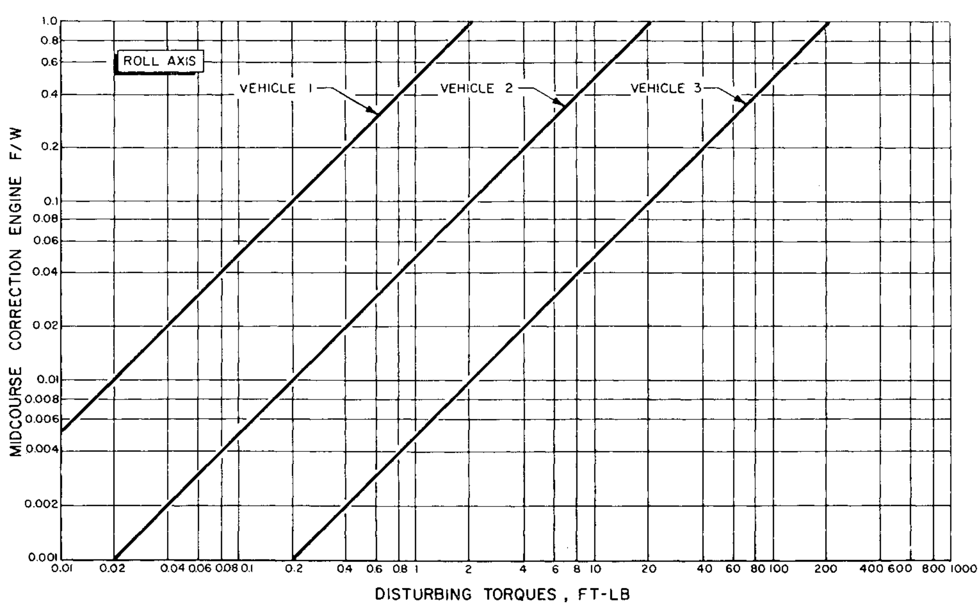

Fig. 9 **Midcourse correction engine disturbances.**

Table 3 Thrust level

	Vehicle no.		
	1	2	3
Thrust,[a] lb			
Cold gas	0.90	9.0	65
Bipropellant	0.47	4.5	42
Monopropellant	0.11	1.1	10

[a] Total of two nozzles.

Table 4 Summary of impulse[a] requirements

	Vehicle no.		
	1	2	3
Acquisition	26.4	281	2848
Rotation	48.7	1067	22,980
Midcourse correction	21.0	131	690
Position stabilization			
Cold gas	16.5	142	1136
Monopropellant	33.7	199	918
Bipropellant	13.0	95	699
Total impulse, lb-sec			
Cold gas	112.6	1621	27,654
Monopropellant	129.8	1678	27,436
Bipropellant	109.1	1574	27,217

[a] All values in lbs-sec.

2) Three-axis stabilization about a deadband of 0.15° for a period of 30 min.

3) Roll-axis control for 5 min during midcourse correction engine firing ($F/W = 0.05$, $\delta = 0.15°$).

4) The vehicle is rotated back to its original position after the midcourse correction maneuver is completed and the deadband stabilization is resumed for the remainder of the trip or until the next correction.

5) There are five midcourse corrections with two complete rotational maneuvers per correction.

Before midcourse correction, the vehicle is in the position-stabilization mode with a deadband of 1.5°. Just prior to the midcourse maneuver, the deadband is reduced to 0.15° where it is held for 30 min; during this time the attitude control supplies the required impulse for three-axis stabilization. The solar disturbing torques are still acting on the vehicle, but the limit cycle will be undisturbed because the thrust levels were sized on the basis of the 1.5° deadband. If a thrust level is to be used to insure optimum disturbed limit cycle operation, it would be related to the existing values by the square root of the ratio of the two deadbands, i.e., $1/10^{1/2}$ in the present case. It does not seem practical, however, to have a new set of reaction jets to accomplish just this control maneuver, especially since it is of such short duration.

To compensate for the variation of the solar torques, their maximum values were assumed for the entire mission in calculating the impulse requirements. The results are given in Table 4. The propellant weights that will be utilized during the sample missions depend on the specific impulse of the particular propellant combination ($W_p = I/I_s$). Typical values of specific impulse for the three propellant combinations are shown in Table 5. The propellant weights can be easily obtained from these values.

If a system was actually designed for the missions under discussion, additional propellant would have to be provided to insure success, because deviations from nominal conditions are likely to occur.

Factors that must be taken into account are meteoritic impacts,[#] excessive acquisition velocity errors, electronic noise effects on system operation (detailed discussion is given in Ref. 8), variations in solar torques, and equipment failure.

Evaluation of all these factors is beyond the scope of this paper. It will suffice to state that the most serious case is presented when equipment fails with the propellant valves stuck open; a factor of 3 in propellant weight may be necessary unless backup valves are provided which can close when this failure occurs. This factor is required because for every pound of propellant lost, 2 lb are required—1 lb to neutralize the imparted disturbance caused by the leakage, and 1 lb for normal control operations.

The factor of 3 was adopted for the small vehicle only; for the larger vehicles smaller factors were adopted for reasons of additional weight penalty as a function of attained reliability. The detailed discussion is given in Ref. 1 and the final results are given in Table 6.

[#] Several authors report that this does not constitute a major source of disturbance.[9]

Table 5 Delivered specific impulse[a]

Cold gas (GN_2), sec	70
Monopropellant (N_2H_4), sec	230
Bipropellant (N_2O_4/N_2H_4), sec.	300

[a] Additional information may be found in Ref. 6.

Summary and Conclusions

The foregoing discussion clearly indicates that the thrust level of an all-jet attitude control system is integrally connected with the guidance requirements and propellants used. It has also been indicated how careful analysis of each maneuver can result in a single thrust level system without penalizing the impulse requirements. Thrust levels based on the disturbed limit cycle concept are propellant dependent by virtue of their different minimum impulse capabilities.

The numerical results stem from the vehicles, mission, and requirements assumed; variation in any of these items will alter them. Variations in the magnitude of the disturbing torques caused by solar pressure will directly affect the propellant consumption; variations in the minimum impulse capability would affect only the thrust levels, which would vary in accordance with the constraint that the impulse is fixed by the solar torques. If the thrust is not varied the limit cycle may become undisturbed and the impulse requirements will increase accordingly. The minimum impulse values used here are typical and attainable by presently existing systems.

Table 6 Final system summary

	Vehicle no.		
	1	2	3
Single nozzle thrust, lb			
Cold gas	0.45	4.5	32.5
Bipropellant	0.235	2.25	21
Monopropellant	0.055	0.55	5
Maximum firing time, sec			
Cold gas	12.5	13.6	20.8
Bipropellant	25	27.3	31.4
Monopropellant	58.14	67.8	73
Maximum number of cycles (roll axis)			
Cold gas	4548	3007	1586
Bipropellant	4548	2963	1549
Monopropellant	5317	3300	1603
Pulsing frequency (all systems)			
Maximum, cps	3.0	1.9	1.0
Minimum, cycles per day	2.79	1.08	0.5
Total propellant weight, lb			
Cold gas	6.0	36	494
Bipropellant	1.32	8.4	115
Monopropellant	1.72	11	150

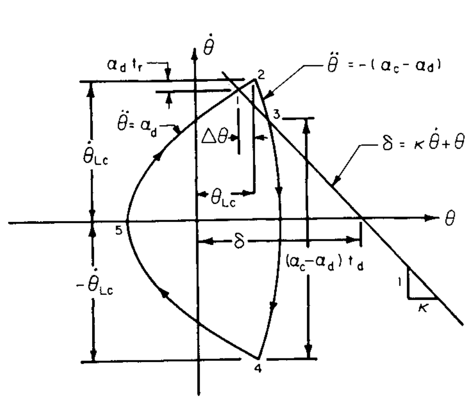

Fig. 11 Disturbed limit cycle.

Finally, if the deadband magnitude is decreased it will change the type and affect the frequency of the limit cycle; hence, propellant consumption will increase.

The unqualified success of the Mariner II attitude control system is indicative of the applicability of jet systems in controlling spacecraft. The low propellant consumption of 2.2 lb of gaseous nitrogen for 110 days serves as an additional indication of the role of jets in future space missions.[7]

Appendix: Disturbed Limit Cycle Equations

The limit cycle velocity $\dot{\theta}_{Lc}$ from Fig. 11 is

$$\dot{\theta}_{Lc} = \dot{\theta}_2 = -\dot{\theta}_4 \qquad \theta_2 = \theta_4 = \theta_{Lc} \tag{A1}$$

$$\dot{\theta}_4 = \dot{\theta}_3 - (\alpha_c - \alpha_d)t_d \tag{A2}$$

$$\dot{\theta}_3 = (\delta - \theta_3)/k \tag{A3}$$

$$\theta_3 = \theta_{Lc} - \frac{(\dot{\theta}_3{}^2 - \dot{\theta}_{Lc}{}^2)}{2(\alpha_c - \alpha_d)} \tag{A4}$$

Simultaneous solution of Eqs. (A3) and (A4) gives

$$\dot{\theta}_3 = k(\alpha_c - \alpha_d) - [k^2(\alpha_c - \alpha_d)^2 - 2(\alpha_c - \alpha_d)(\delta - \theta_{Lc}) + \dot{\theta}_{Lc}{}^2]^{1/2} \tag{A5}$$

Substituting Eqs. (A5) and (A1) into Eq. (A2) obtains

$$-\dot{\theta}_{Lc} = k(\alpha_c - \alpha_d) - t_d(\alpha_c - \alpha_d) - [k^2(\alpha_c - \alpha_d)^2 - 2(\alpha_c - \alpha_d)(\delta - \theta_{Lc}) + \dot{\theta}_{Lc}{}^2]^{1/2} \tag{A6}$$

$$\begin{aligned}\delta - \theta_{Lc} &= k\dot{\theta}_1 - \dot{\theta}_1 t_r - \alpha_d t_r{}^2/2 \\ &= (\dot{\theta}_{Lc} - \alpha_d t_r)(k - t_r) - \alpha_d t_r{}^2/2\end{aligned} \tag{A7}$$

Substituting Eq. (A7) into Eq. (A6) and solving for $\dot{\theta}_{Lc}$ obtains

$$\dot{\theta}_{Lc} = \frac{1}{2}\left\{\frac{\alpha_c t_d(2k - t_d)}{2k - t_r - t_d} + \alpha_d(t_r - t_d)\right\} \tag{A8}$$

For $k \gg t_d$ and $k \gg t_r$, Eq. (A8) becomes

$$\dot{\theta}_{Lc} \approx \tfrac{1}{2}\{\alpha_c t_d + \alpha_d(t_r - t_d)\} \tag{A9}$$

and for low disturbances ($\alpha_d \ll 1$):

$$\dot{\theta}_{Lc} \approx \tfrac{1}{2}\alpha_c t_d \tag{A10}$$

Time "on"

$$t_{\text{on}} = 2\dot{\theta}_{Lc}/(\alpha_c - \alpha_d) \approx t_d \tag{A11}$$

Time "off"

$$t_{\text{off}} = 2\dot{\theta}_{Lc}/\alpha_c = (r - 1)t_{\text{on}} \approx (r - 1)t_d \tag{A12}$$

Time "off" to time "on" ratio

$$f = t_{\text{off}}/t_{\text{on}} = r - 1 \tag{A13}$$

Propellant consumption

$$W_p = (T_D/I_s L_f)X \tag{A14}$$

Cycling frequency

$$\bar{f} = \frac{1}{t_{\text{on}}(1 + f)} = \frac{1}{t_{\text{on}} r} \approx \frac{1}{t_d r} \tag{A15}$$

If the disturbing torque reverses the vehicle velocity at one end of the deadband, the following is obtained:

$$\theta_5 = -\delta \qquad \dot{\theta}_5 = 0 \tag{A16}$$

The limit cycle velocity is then dictated by the disturbances and is given by

$$\dot{\theta}_{Lc} = \alpha_d \left\{t_r - k + \left[k^2 + 4\frac{\delta}{\alpha_d}\right]^{1/2}\right\} \tag{A17}$$

For low disturbing torques (A17) becomes

$$\dot{\theta}_{Lc} = 2\alpha_d(\delta/\alpha_d)^{1/2} \tag{A18}$$

The velocity as given by Eq. (A18) must be matched by the control. Combining Eq. (A18) with Eq. (A10) the required torque ratio r to insure such an operation is

$$r = 4(\delta/\alpha_d)^{1/2}/t_d \tag{A19}$$

The control thrust for this operation is the maximum possible for a disturbed limit-cycle operation without any impulse penalty, and is given by

$$F_c = 4(T_D J\delta)^{1/2}/L_f t_d \tag{A20}$$

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